PMM U.S.S.R.,Vol.52,No.2,pp.141-146,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

SEMICONTINUOUS SOLUTIONS OF HAMILTON-JACOBI EQUATIONS*

I. ROZYEV and A.I. SUBBOTIN

Generalized (viscosity) solutions of Hamilton-Jacobi and Bellman-Isaacs equations are defined by means of pairs of differential inequalities. This type of solution exists, is unique and in the case of the Bellman-Isaacs equation, is identical with the value function of the corresponding differential game. Unlike the published literature (/1-8/ etc.), in which continuous solutions were considered, differential games are considered here with semicontinuous payoff functions and correspondingly semicontinuous solutions. Definitions are introduced for these solutions and existence and uniqueness theorems are proved.

Problems in which the value function (Bellman function) may fail to be continuous are well-known. For example, in a differential game of pursuit-evasion the payoff function, defined as the time to capture, is lower semicontinuous and the corresponding value function is also lower semicontinuous. In particular, the Bellman function of the optimum response problem is lower semicontinuous; its properties have been studied by many authors, and differential relations have been derived /9, 10/ that represent the optimality principle in the response problem.

The theory of differential games provided a suitable framework for studying many questions in the general theory of Hamilton-Jacobi equations. The aim of this paper is to develop an apparatus of differential inequalities and define generalized (viscosity) solutions for the case in which these solutions are semicontinuous. To fix our ideas, a differential game with fixed terminal time is considered. Results are then formulated for the game of pursuit-evasion.

1. A differential game with fixed terminal time.

Let the motion of a controlled system be described by the equation

$$x^{\cdot} = f(t, x, u, v), \quad u \in P, \; v \in Q; \; f\colon T \times R^n \times P \times Q \to R^n \tag{1.1}$$

where P and Q are compact subsets of spaces R^p and R^q, respectively. The function f is continuous and satisfies a Lipshitz condition with respect to x. We shall assume that the game begins at a time $t_0 \in T = [0, \theta]$ and ends at a time $t = \theta$. It is also assumed that the following condition holds:

$$\min_{u\in P} \max_{v\in Q} \langle s, f(t, x, u, v)\rangle = \max_{v\in Q} \min_{u\in P} \langle s, f(t, x, u, v)\rangle, \quad (t, x, s) \in T \times R^n \times R^n \tag{1.2}$$

($\langle a, b\rangle$ is the scalar product of vectors a and b).

The differential game is considered in the class of positional strategies. The payoff functional is defined by

$$\gamma(x(\cdot)) = \sigma(x(\theta)); \; x(\cdot)\colon [t_0, \theta] \to R^n, \; \sigma\colon R^n \to R \tag{1.3}$$

where $x(\cdot)$ is the actually realized motion of system (1.1) and σ is a give function. A formalization of positional differential games may be found, e.g., in /11-13/. If σ is a (lower or upper) semicontinuous function, then for any initial position $(t_0, x_0) \in T \times R^n$ there exists a game value $\omega(t_0, x_0)$. If the value function $(t, x) \to \omega(t, x)$ is differentiable in some domain, then in that domain it will satisfy the Bellman-Isaacs equation

$$\partial\omega/\partial t + H(t, x, \partial\omega/\partial x) = 0 \tag{1.4}$$

The value function satisfies the boundary condition

$$\omega(\theta, x) = \sigma(x) \tag{1.5}$$

In Eq.(1.4) $H(t, x, s)$ is the Hamiltonian of system (1.1), defined by the equality

*Prikl.Matem.Mekhan.,52,2,179-185,1988

$$H(t, x, s) = \min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v)\rangle \tag{1.6}$$

Below we shall define the notation of a generalized solution of problem (1.4), (1.5) in the case when the payoff function, and hence also the value function, are semicontinuous. The constructions we propose develop results obtained in /1-8/ for continuous or Lipschitzian functions σ and ω.

Let $\omega: T \times R^n \to R$ be some functions. Choose a point $(t, x) \in T^\circ \times R^n$ $(T^\circ = [0, \theta))$ and a vector $h \in R^n$. We introduce the following notation·

$$\partial_- \omega(t, x) \mid (1, h) = \varliminf_{\delta \downarrow 0, \|g\| \downarrow 0} \Omega \quad \partial_+ \omega(t, x) \mid (1, h) = \varlimsup_{\delta \downarrow 0, \|g\| \downarrow 0} \Omega \tag{1.7}$$

$$(\Omega = [\omega(t + \delta, x + \delta h + \delta g) - \omega(t, x)]\delta^{-1})$$

$$G_\omega^-(t, x) = \{h \in R^n: \partial_- \omega(t, x) \mid (1, h) \leqslant 0\} \tag{1.8}$$

$$G_\omega^+(t, x) = \{h \in R^n: \partial_+ \omega(t, x) \mid (1, h) \geqslant 0\}$$

$$\rho_\omega^-(t, x \mid l) = \sup_h \langle h, l\rangle, \quad h \in G_\omega^-(t, x) \tag{1.9}$$

$$\rho_\omega^+(t, x \mid l) = \inf_h \langle h, l\rangle, \quad h \in G_\omega^+(t, x)$$

Consider the differential inclusion relations

$$x^\cdot(t) \in F_1(t, x(t), v), \quad x^\cdot(t) \in F_2(t, x(t), u) \tag{1.10}$$

$$F_1(t, x, v) = \operatorname{co}\{f(t, x, u, v): u \in P\} \tag{1.11}$$

$$F_2(t, x, u) = \operatorname{co}\{f(t, x, u, v): v \in Q\}$$

($\operatorname{co}\{\cdot\}$ denotes the convex hull of a set).

The set of absolutely continuous functions $x(\cdot): T \to R^n$ satisfying the first (second) differential inclusion and also the condition $x(t_0) = x_0$ will be denoted by $X_1(t_0, x_0, v)$ $(X_2(t_0, x_0, u))$.

Definition 1.1. A supersolution of problem (1.4), (1.5) is a lower semicontinuous function $\omega: T \times R^n \to R$ satisfying the conditions

$$\rho_\omega^-(t, x \mid l) \geqslant H(t, x, l), \quad (t, x, l) \in T^\circ \times R^n \times R^n \tag{1.12}$$

$$\omega(\theta, x) \geqslant \sigma(x), \quad x \in R^n \tag{1.13}$$

Definition 1.2. A subsolution of problem (1.4), (1.5) is an upper semicontinuous function $\omega: T \times R^n \to R$ satisfying the conditions

$$\rho_\omega^+(t, x \mid l) \leqslant H(t, x, l), \quad (t, x, l) \in T^\circ \times R^n \times R^n \tag{1.14}$$

$$\omega(\theta, x) \leqslant \sigma(x), \quad x \in R^n \tag{1.15}$$

Definition 1.3. A function $\omega^0: T \times R^n \to R$ is called a generalized solution of problem (1.4), (1.5) if there exist sequences of supersolutions ω^k and subsolutions ω_k $(k = 1, 2, \ldots)$, converging pointwise to ω^0.

Inequality (1.12) means that ω has the u-stability property /6, 11, 13/. This property may also be defined by the equivalent inequalities

$$\sup_{(t_1, x_1, t_2, v)} \inf_{x(\cdot) \in X_1(t_1, x_1, v)} [\omega(t_2, x(t_2)) - \omega(t_1, x_1)] \leqslant 0 \tag{1.16}$$

$$(t_1, x_1, v) \in T^\circ \times R^n \times Q, \quad t_2 \in (t_1, \theta]$$

$$\max_{v \in Q} \min_{h \in F_1(t, x, v)} \partial_- \omega(t, x) \mid (1, h) \leqslant 0, \quad (t, x) \in T^\circ \times R^n \tag{1.17}$$

Similarly, inequality (1.14) means that ω has the v-stability property and may be replaced by either of the two equivalent inequalities

$$\inf_{(t_1, x_1, t_2, u)} \sup_{x(\cdot) \in X_2(t_1, x_1, u)} [\omega(t_2, x(t_2)) - \omega(t_1, x_1)] \geqslant 0 \tag{1.18}$$

$$(t_1, x_1, u) \in T^\circ \times R^n \times P, \quad t_2 \in (t_1, \theta]$$

$$\min_{u \in P} \max_{h \in F_2(t, x, u)} \partial_+ \omega(t, x) \mid (1, h) \geqslant 0, \quad (t, x) \in T^\circ \times R^n \tag{1.19}$$

The following existence and uniqueness theorems hold for the generalized solutions.

Theorem 1.1. Let the Hamiltonian H satisfy equality (1.6), where f satisfies the conditions enumerated above. If there exists a generalized solution of problem (1.4), (1.5), then it is unique and identical with the value function of the positional differential game (1.1), (1.3).

Theorem 1.2. Under the assumptions of Theorem 1.1, let the function σ be lower (upper) semicontinuous and bounded. Then a generalized solution of problem (1.4), (1.5) exists, is unique, is identical with the value function of the differential game (1.1), (1.3) and is a supersolution (subsolution) of problem (1.4), (1.5).

One result of Theorem 1.2 is that if σ is continuous, then a solution of problem (1.4), (1.5) is also a sub- and supersolution of the problem. The proofs of Theorems 1.1 and 1.2 rely on an extremal construction evolved in the theory of positional differential games /11, 13/.

2. Stability properties and the extremal construction. As observed above, the properties of u- and v-stability may be defined differently. This statement will now be elucidated.

Lemma 2.1. If ω is a lower (upper) semicontinuous function, then conditions (1.12), (1.16), (1.17), ((1.14), (1.18), (1.19)) are equivalent.

The proof is essentially the same as the proofs of analogous propositions in /6, 8, 14/*. (*See also Kh.G. Guseinov, On a definition of the stability property of the value function of a differential game by inequalities, Baku, 1986. Dep. at VINITI 04.04.86, No.2408-B). It uses Lemma 2 of /15/.

Theorem 1.1 can be proved along the same lines as Theorem 5.3 in /6/.

Proof of Theorem 1.2. Consider the case of a lower semicontinuous function σ. Put

$$E = \operatorname{epi} \sigma = \{(r, x) \in R \times R^n: r \geqslant \sigma(x)\}$$
$$R_\alpha(x) = \{r \in R: \operatorname{dist}[(r, x), E] \geqslant \alpha\}$$

$$\sigma^\alpha(x) = \max R_\alpha(x), \quad \alpha \in (0, 1]$$
$$\operatorname{dist}[(r, x), E] = \min [(r - r_*)^2 + \|x - x_*\|^2]^{1/2}, \quad (r_*, x_*) \in E$$

The function σ^α has the following properties for any $\alpha \in (0, 1]$ and $x \in R^n$:

$$\varlimsup_{y \to x} \sigma^\alpha(y) \leqslant \sigma^\alpha(x), \quad \lim_{\alpha \downarrow 0,\, \overline{y \to x}} \sigma^\alpha(y) = \sigma(x) \tag{2.1}$$
$$\sigma^\alpha(x) \leqslant \sigma(x) - \alpha$$
$$\sup_{\alpha, x} |\sigma^\alpha(x)| \leqslant l < \infty; \quad \sigma^{\alpha_1}(x) \geqslant \sigma^{\alpha_2}(x), \quad \alpha_1 \leqslant \alpha_2$$

Let Ω_1^α (Ω_2^α) denotes the set of functions ω: $T \times R^n \to [-l, l]$ satisfying condition (1.16) ((1.18)) and the inequality $\omega(\theta, x) \geqslant \sigma^\alpha(x)$ $(\omega(\theta, x) \leqslant \sigma^\alpha(x))$ for $x \in R^n$.

Consider the function

$$\omega^\alpha(t, x) = \inf \omega(t, x), \quad \omega \in \Omega_1^\alpha$$

As shown in /16/, $\omega^\alpha \in \Omega_1^\alpha \cap \Omega_2^\alpha$. From the last property in (2.1) it follows that

$$\Omega_1^{\alpha_1} \subset \Omega_2^{\alpha_2}, \quad \omega^{\alpha_2}(t, x) \leqslant \omega^{\alpha_1}(t, x) \quad \textbf{for} \quad \alpha_1 \leqslant \alpha_2 \tag{2.2}$$

It can be shown /16/ that the function ω° defined by

$$\omega^\circ(t, x) = \lim_{\alpha \downarrow 0,\, \overline{(\tau, y) \to (t, x)}} \omega^\alpha(\tau, y) \tag{2.3}$$

satisfies Definition 1.1, i.e., it is a supersolution of problem (1.4), (1.5).

We define the function

$$\omega_+^\alpha(t, x) = \varlimsup_{(\tau, y) \to (t, x)} \omega^\alpha(\tau, y) \tag{2.4}$$

It follows immediately from the definition that this function is upper semicontinuous. It satisfies inequality (1.18) (see the analogous assertion in /16/). Using the first and third of properties (2.1), we obtain the limit $\omega_+^\alpha(\theta, x) \leqslant \sigma(x)$, $x \in R^n$. Thus the function ω_+^α is a subsolution of problem (1.4), (1.5).

Consider the sequence of functions $\omega_k = \omega_+^{\alpha_k}$, $k = 1, 2, \ldots$, $\alpha_{k+1} < \alpha_k$, $\alpha_k \to 0$ as $k \to \infty$. It follows from (2.2) that $\omega_k(t, x)$ is an increasing sequence, and so the limit $\lim \omega_k(t, x) = \omega_+(t, x)$ exists as $k \to \infty$. Using the fact that ω° is a supersolution and ω_+^α a subsolution of problem (1.4), (1.5), one can show that $\omega^\circ(t, x) \geqslant \omega_+^\alpha(t, x)$ /6, 16/. Therefore $\omega^\circ(t, x) \geqslant \omega_+(t, x)$. On the other hand, by (2.3) and (2.4) we have $\omega^\circ(t, x) \leqslant \omega_+(t, x)$. Consequently,

$\omega^\circ(t, x) = \omega_+(t, x)$. Thus we obtain $\omega^\circ(t, x) = \lim \omega_k(t, x)$ as $k \to \infty$, where ω° is a supersolution and ω_k $(k = 1, 2, \ldots)$ a sequence of subsolutions. This proves Theorem 1.2 for the case of a lower semicontinuous function σ. The proof for upper semicontinuous σ is analogous.

3. Game-theoretic fast-response problem. We will present some results relating to the problem of (M, N)-approach /11, 13/. Let the motion of a controlled system be described by Eq.(1.1). Suppose we are given sets $M \subset N \subset [0, \theta] \times R^n$, where $\theta_* < \theta$. The set M is the objective set for the first player, whose goal is to make the point $(t, x(t))$ reach M in minimum time. The set N is a phase constraint, requiring that, on its way from the initial position to M, the condition $(t, x(t)) \in N$ is satisfied.

The payoff functional $\tau(x(\cdot))$ is defined as follows. Let $x(\cdot)\colon [t_0, \theta] \to R^n$ be a continuous function. We put

$$T(x(\cdot)) = \{\tau \in [t_0, \theta]\colon (\tau, x(\tau)) \in M,\ (t, x(t)) \in N,\ t_0 \leqslant t \leqslant \tau\} \tag{3.1}$$

$$\tau(x(\cdot)) = \begin{cases} \inf T(x(\cdot)), & T(x(\cdot)) \neq \varnothing \\ \theta, & T(x(\cdot)) = \varnothing \end{cases}$$

Instead of θ we could have taken any number $\theta^* > \theta_*$, in particular, the "improper" number $+\infty$.

If the sets M and N are closed, it is known that the differential game (1.1), (3.1) for any initial position $(t_0, x_0) \in T \times R^n$ has a value $\omega^\circ(t_0, x_0)$. If the value function ω° is differentiable in some open domain $O \subset T \times R^n$, it satisfies Eq.(1.4) in that domain. It follows from the definition of the payoff function (3.1) and the value function that

$$\omega^\circ(t, x) \geqslant t,\quad \forall (t, x) \in T \times R^n \tag{3.2}$$

$$\{(t, x)\colon \omega^\circ(t, x) = t\} = M,\quad \{(t, x)\colon \omega^0(t, x) \leqslant \theta_*\} \subset N$$

It is also known that if the sets M and N are closed then ω° is lower semicontinuous.

In order to define the value function as a generalized solution of problem (1.4), (3.2), we propose once more to replace Eq.(1.4) by a pair of differential inequalities. Note that the following definitions do not require M and N to be closed sets.

Definition 3.1. A supersolution of problem (1.4), (3.2) is a lower semicontinuous function $\omega\colon T \times R^n \to T$ satisfying the conditions

$$\rho_\omega^-(t, x \mid l) \geqslant H(t, x, l),\quad (t, x) \in (T^\circ \times R^n) \setminus M_c,\ l \in R^n$$

$$\omega(t, x) \geqslant t,\quad (t, x) \in T \times R^n$$

$$\{(t, x) \in T \times R^n\colon \omega(t, x) \leqslant \theta_*\} \subset N_c$$

where M_c and N_c are certain closed sets such that $M_c \subset M$, $N_c \subset N$.

Definition 3.2. A subsolution of problem (1.4), (3.2) is an upper semicontinuous function $\omega\colon T \times R^n \to T$ satisfying the conditions

$$\rho_\omega^+(t, x \mid l) \leqslant H(t, x, l),\quad (t, x) \in N^\circ$$

$$\omega(t, x) \geqslant t,\quad (t, x) \in T \times R^n$$

$$\{(t, x) \in T \times R^n : \omega(t, x) = t\} \supset M^\circ$$

where M° and N° are certain open sets containing M and N, respectively.

Definition 3.3. A function $\omega^\circ : T \times R^n \to T$ is called a generalized solution of problem (1.4), (3.2) if there exist sequences of supersolutions ω^k and subsolutions ω_k $(k = 1, 2, \ldots)$ that converge pointwise to ω°.

Analogues of Theorem 1.1 and 1.2 hold for the game-theoretic fast-response problem.

Theorem 3.1. Let the Hamiltonian H be defined by (1.6) and assume that the function f satisfies the conditions enumerated in Sect.1. If there exists a generalized solution to problem (1.4), (3.2), it is unique and identical with the value function of the differential game (1.1), (3.1), i.e., with the function defining the optimum outcome in the game-theoretic fast-response problem.

Theorem 3.2. Under the assumptions of Theorem 3.1, let the sets M and N be closed. Then there exists a generalized solution of problem (1.4), (3.2), it is unique, is identical with the value function of the differential game (1.1), (3.1) and is a supersolution of problem (1.4), (3.2).

The proofs, which are for the most part similar to those of Theorems 1.1 and 1.2, will be omitted.

Note that the usual procedure in the fast-response problem is to define the objective functional not as the time $\tau(x(\cdot))$ (3.1) at which the motion reaches the target set but as the time $t(x(\cdot))$ elapsing up to that event. Clearly, $t(x(\cdot)) = \tau(x(\cdot)) - t_0$, $w^\circ(t_0, x_0) = \omega^\circ(t_0, x_0) - t_0$, where w° is the value function in the fast-response with objective functional $t(x(\cdot))$. The relationships (3.2) may be rewritten in terms of w°:

$$w^\circ(t, x) \geqslant 0, \quad \forall (t, x) \in T \times R^n$$
$$\{(t, x)\colon w^\circ(t, x) = 0\} = M, \quad \{(t, x)\colon w^\circ(t, x) \leqslant \theta_* - t\} \subset N \tag{3.3}$$

If w° is differentiable in some open domain $O \subset T \times R^n$, then in that domain is satisfies the equation

$$\partial w^\circ/\partial t + H(t, x, \partial w^\circ/\partial x) = -1 \tag{3.4}$$

It is also obvious how Definitions 3.1-3.3 must be modified when problem (1.4), (3.2) is replaced by (3.3), (3.4). Under the assumptions of Theorem 3.2, a generalized solution of problem (3.3), (3.4) exists, is unique and identical with the value function w° in the fast-response problem. Note that here the differential inequalities may be written as follows (see (1.7), (1.11):

$$\max_{v \in Q} \min_{h \in F_1(t, x, v)} \partial w(t, x) \mid (1, h) \leqslant -1$$
$$\min_{u \in P} \max_{h \in F_2(t, x, u)} \partial w(t, x) \mid (1, h) \geqslant -1$$

4. Conclusion.

The essential point in our definition of generalized solutions for equating of type (1.4) is as follows. A generalized solution is a minorant for supersolutions and simultaneously a majorant for subsolutions. In the definitions of supersolutions and subsolutions, the original Eq.(1.4) is replaced by a differential inequality (1.12) or (1.14), respectively. These inequalities state that the value function is u- or v-stable and may be written in various equivalent forms.

In this paper we have investigated two problems, defining generalized solutions for them and proving existence and uniqueness theorems. Analogous results hold for problems of other types, e.g., for differential games with payoff functions of the form

$$\sigma(x(\theta)) + \int_{t_0}^{\theta} f_0(t, x(t), u(t), v(t))\, dt, \quad \min_{t_0 \leqslant t \leqslant \theta} \rho(t, x(t))$$

Some of the assumptions adopted above are not essential. In particular, condition (1.2) can be dropped, but then inequalities (1.16) and (1.19) must be modified, as was done in /1, 6, 8/, where problems with continuous payoff functionals were considered.

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Translated by D.L.

PMM U.S.S.R.,Vol.52,No.2,pp.146-151,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

CONDITIONS FOR A SUM OF FORMS TO BE OF FIXED SIGN AND FOR STABILITY OF MOTION ON MANIFOLDS*

A.B. AMINOV and T.K. SIRAZETDINOV

Lyapunov's corollary of the Stability Theorem /1/, a special case of which is Routh's theorem on the stability of the steady motion of a system with cyclic coordinates, provides a point of departure for the investigation conducted in this paper of the stability of motion on manifolds, particularly those defined by the integrals of the equations of the perturbed motion. Sufficient conditions are obtained for a sum of forms to be positive- or negative-definite and for the motion of polynomial systems to be stable on these manifolds.

1. Given a sum of forms

$$F(\mathbf{x})=\sum_{s=2}^{2q} X^{(s)}(\mathbf{x}, A_{i_1\ldots i_s}),\quad \mathbf{x}=(x_1,\ldots,x_n)\in R_x^{\,n} \tag{1.1}$$

and a manifold M defined by equalities

$$F_r(\mathbf{x})=\sum_{s=1}^{p} X_r^{(s)}(\mathbf{x}, B_{ri_1\ldots i_s})=0,\quad r=1,2,\ldots,m;\, m<n,\, p<q \tag{1.2}$$

where $X^{(s)}(\mathbf{x}, A_{i_1\ldots i_s})$, $X_r^{(s)}(\mathbf{x}, B_{ri_1\ldots i_s})$ are multilinear forms of degree s, of the form

$$X^{(s)}(x, A_{i_1\ldots i_s})=\sum_{i_1=1}^{n}\cdots\sum_{i_s=i_{s-1}}^{n} A_{i_1\ldots i_s}x_{i_1}\ldots x_{i_s}$$

$A_{i_1,\ldots,i_s}$, $B_{ri_1,\ldots,i_s}$ are real numbers, p, q, s, m, n are positive integers, and $R_x^{\,n}$ is Euclidean n-space. Like terms in the forms are reduced and the terms are assumed to be lexicographically ordered.

We shall determine the sufficient conditions for functions (1.1) to be positive- or negative-definite under constraints (1.2).

Let R^N_y denote the Euclidean space of vectors $\mathbf{y}=(y_1,\ldots,y_N)$ and $\Phi: R_x^{\,n}\to R_y^{\,N}$ the mapping defined as follows:

$$\begin{aligned}&y_1=x_1^{\,q},\quad y_2=x_1^{\,q-1}x_2,\quad \ldots,\quad y_n=x_1^{\,q-1}x_n\\&y_{n+1}=x_1^{q-2}x_2^{\,2},\quad y_{n+2}=x_1^{q-2}x_2x_3,\ldots,y_{N-n+1}=x_1,\\&y_{N-n+2}=x_2,\ldots,y_N=x_n\end{aligned} \tag{1.3}$$

i.e., $y_j=x_{i_1}x_{i_2}\ldots x_{i_s}$ and $j\rightleftarrows i_1i_2\ldots i_s$, where $i_1\leqslant i_2\leqslant\ldots\leqslant i_s$, $i_1, i_2,\ldots,i_s=1,2,\ldots,n;\, j=1,2,\ldots,N$.

Lemma 1.1. A sum of forms $F(\mathbf{x})$ (1.1) defined in $R_x^{\,n}$ is mapped by Φ (1.3) into a certain quadratic form (q.f.)

Prikl.Matem.Mekhan.,52,2,186-193,1988